Plastic Zones at the Tips of Inclined Cracks in Glassy Polymers under Small Scale Yielding

E. E. GDOUTOS, *Chair of Applied Mechanics, Democritus University of Thrace, Xanthi, Greece*

Synopsis

A thorough study of the shape and size of plastic zones developed around cracks subjected to combined opening-mode and sliding-mode loading conditions in glassy polymers under small scale yielding was undertaken. Two pressure-modified von Mises yield criteria which take into account the characteristic behavior of glassy polymers expressed by the dependence of their yield locus on the hydrostatic stress component and the difference in their tensile and compressive yield stresses were used. The case of an infinite plate subjected to a uniaxial uniform stress at an arbitrary inclination with respect to the axis of the crack was considered. From the whole study useful results concerning the dependence of the shape and size of plastic zones on the crack inclination angle, the Poisson's ratio, and the ratio of the compressive to tensile yield stress of the plate were derived.

INTRODUCTION

In a previous paper[1] the plastic zones developed at the tips of cracks in plates made of glassy polymers and subjected to opening-mode loading conditions were studied. Two pressure-modified von Mises yield criteria which take into account the particular behavior of glassy polymers expressed by the dependence of their yield locus on the hydrostatic stress component and the difference in their tensile and compressive yield stresses were used. It was found that the plastic zones predicted by both these criteria are larger than those obtained by the von Mises criterion and that they increase with the Poisson's ratio of the cracked plate.

In-plane cracks, however, are generally subjected to two basic modes of crack extension: the opening-mode encountered in symmetrical tension and bending and the sliding-mode associated with shear normal to the leading edge of the crack. Cracks in structures are generally randomly oriented with respect to applied loads and, therefore, they are subjected to both opening-mode and sliding-mode loading conditions. Most of the literature in crack problems concerns investigations dealing with opening-mode conditions. This is mainly due to the simplicity in analytical computations and experimentation involved in opening-mode crack problems.

Determination of plastic zones around crack tips is of great importance in fracture mechanics. The extend of the plastic zone influences the crack propagation velocities which are normally much less for plastic than for elastic strains. Furthermore, small plastic zones allow the application of linear elastic fracture mechanics in crack problems. This theory has successfully been used for the prediction of the failure behavior of engineering components. Pook[2] determined the plastic zones developed in cracked plates under combined opening-mode and sliding-mode loading conditions. He used the near to the crack-tip linear elastic fracture mechanics stress field expressed by the opening-mode K_I and

Journal of Applied Polymer Science, Vol. 27, 879–892 (1982)
 CCC 0021-8995/82/030879-14$01.40

sliding-mode $K_{\rm II}$ stress intensity factors in conjunction with the von Mises yield criterion.

As was discussed in Ref. 1, the yield behavior of polymeric materials cannot adequately be described by the von Mises criterion which is based on the assumptions of the independence of the yield behavior on the hydrostatic component of the stress state and the equality of the yield stresses in tension and compression. Experimental evidence showed that in polymers both these assumptions are critically questioned. Thus, attempts have been made to establish criteria which adequately describe the yield behavior of polymers under multiaxial states of stress. For a review of these works the interesting reader is referred to in Refs. 1, 3 and 4.

In the present paper the results of Ref. 1 concerning an investigation of the plastic zones developed around crack tips in glassy polymers subjected to opening-mode loading conditions are extended to incorporate the more usual case of combined opening-mode and sliding-mode loads. The same yield criteria as in Ref. 1 which take into account the particular behavior of glassy polymers were used. The case of a crack in an infinite isotropic elastic plate subjected to a uniform uniaxial stress at infinity at an arbitrary inclination with respect to the crack axis was considered. The dependence of the shape and size of the plastic zones on the angle of inclination of the load with respect to the crack axis, on the ratio of the compressive to tensile yield stress, and on the Poisson's ratio of the glassy polymer was investigated.

CRACK TIP STRESS FIELD

Consider a crack of length $2a$ in an elastic plate subjected to combined opening-mode and sliding-mode loading. A system of Cartesian coordinates Oxy is related to the plate with its origin O at the crack tip and the x-axis coinciding with the crack line (Fig. 1). For this case the singular stress components σ_x, σ_y, τ_{xy} in the vicinity of the crack tip are given by the following relations[5]:

$$
\begin{aligned}
\sigma_x &= \frac{K_{\rm I}}{(2\pi r)^{1/2}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)-\frac{K_{\rm II}}{(2\pi r)^{1/2}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\
\sigma_y &= \frac{K_{\rm I}}{(2\pi r)^{1/2}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+\frac{K_{\rm II}}{(2\pi r)^{1/2}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\
\tau_{xy} &= \frac{K_{\rm I}}{(2\pi r)^{1/2}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}+\frac{K_{\rm II}}{(2\pi r)^{1/2}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)
\end{aligned}
\tag{1}
$$

The coefficients $K_{\rm I}$ and $K_{\rm II}$ are called opening-mode and sliding-mode stress intensity factors, respectively. They are independent of the coordinates r, θ and they depend on the geometry and loading of the cracked plate. For the particular case of an infinite plate subjected to a uniaxial uniform stress σ_0 at infinity making an angle β with the crack axis $K_{\rm I}$ and $K_{\rm II}$ are given by[5]

$$K_{\rm I} = \sigma(\pi a)^{1/2}\sin^2\beta, \quad K_{\rm II} = \sigma(\pi a)^{1/2}\sin\beta\cos\beta \tag{2}$$

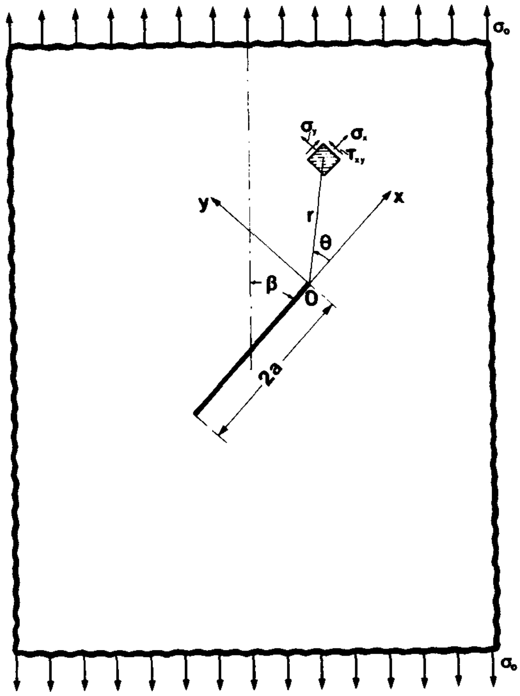

Fig. 1. Geometry of an infinite plate with a crack of length $2a$ subjected to a uniaxial uniform stress σ_0 making an angle β with the crack axis.

FRACTURE CRITERIA IN GLASSY POLYMERS

Two main fracture criteria for the description of the yield behavior of glassy polymers under multiaxial stress states have been introduced. Both these criteria take into account the particular characteristic behavior of polymers whose yield locus depends on the hydrostatic stress component and whose yield stresses in tension and compression are generally different. The first criterion initially proposed by Nadai[6] and applied to the case of glassy polymers by Bauwens[7,8] and Sternstein and Ongchin[9] is expressed by the following relation:

$$(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)^{1/2} + \frac{R-1}{R+1}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{2R}{R+1}\sigma_\tau \tag{3}$$

where

$$R = \sigma_c/\sigma_\tau \tag{4}$$

In these relations σ_c and σ_τ are the yield stresses in compression and tension, respectively. From now on we will refer to this criterion as the pressure-modified octahedral shear stress criterion (PMOSC).

The second criterion was introduced by Raghava, Caddell, and Yeh,[10,11] and it is expressed by the following relation:

$$(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1) + (\sigma_c - \sigma_\tau)(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_c\sigma_\tau \tag{5}$$

In the sequel this criterion will be referred to as the pressure-modified von Mises criterion (PMMC).

Both yield criteria are pressure-dependent and take into account the differences in the yield stresses for tension and compression. By putting $\sigma_c = \sigma_\tau$ both criteria degenerate to the well-known von Mises criterion.

CRACK TIP PLASTIC ZONES

For the determination of the plastic zone at the tip of a crack in a plate subjected to mixed loading conditions under small-scale yielding the singular stress field solution expressed from relations (1) in combination with the above presented yield criteria expressed from relations (3) or (5) were used. The plastic zone surrounding the crack tip is separated from the remaining elastic material by the elastic–plastic boundary.

For the case of thin cracked plates, plane stress conditions characterized by the zeroing of the normal stress σ_z prevail in the vicinity of the crack tip. However, when the plate becomes thicker the stress σ_z is no longer equal to zero

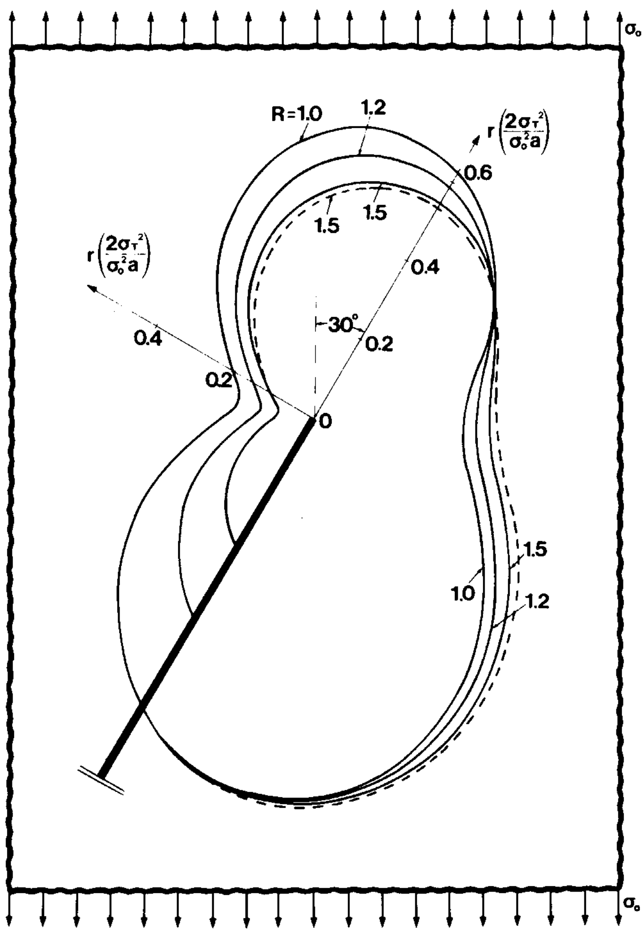

Fig. 2. Elastic–plastic boundaries around the tip of a crack whose axis makes an angle $\beta = 30°$ with respect to the direction of the applied stress σ_0 for various values of the ratio R of the compressive to tensile yield stress of the material. The plate is subjected to plane stress conditions ($\nu = 0$). The continuous curves of the figure correspond to the PMOSC criterion, while the dotted curves correspond to the PMMC criterion.

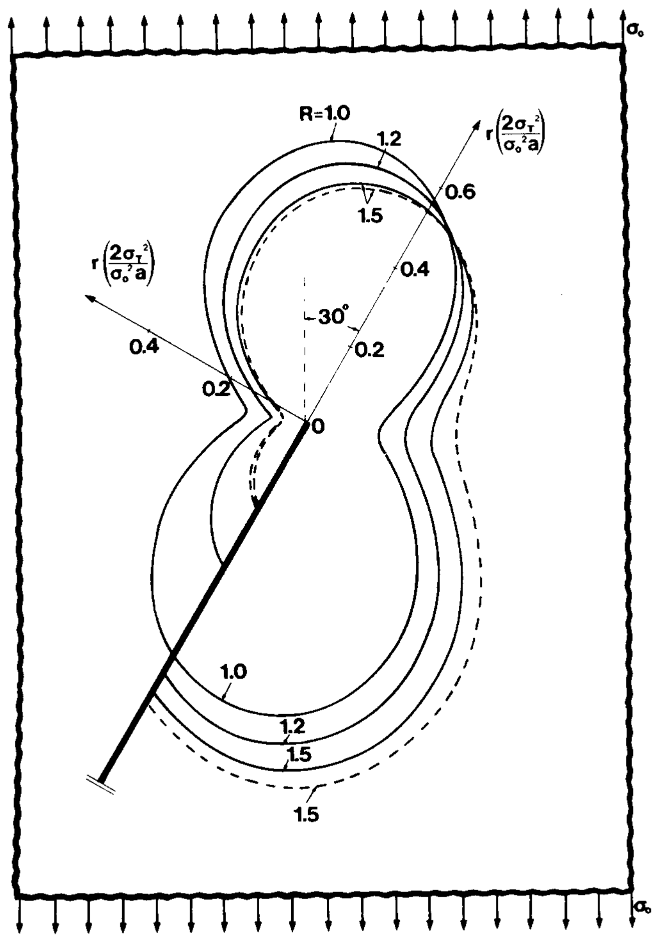

Fig. 3. As in Figure 2 for a cracked plate subjected to plane strain conditions with $\nu = 0.3$.

and plane strain conditions dominate in the area surrounding the crack tip. For this case σ_z is given by

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{6}$$

where ν is the Poisson's ratio of the material of the plate.

Using the identities

$$\begin{gathered} \sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z \\ \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \end{gathered} \tag{7}$$

and taking into account that in our case

$$\tau_{yz} = \tau_{zx} = 0 \tag{8}$$

we obtain after lengthy calculations the following relations:

$$\sigma_1 = \sigma_2 + \sigma_3 = (2/\pi r)^{1/2} (K_{\mathrm{I}} \cos \theta/2 - K_{\mathrm{II}} \sin \theta/2) \tag{9}$$

for plane stress, and

$$\sigma_1 + \sigma_2 + \sigma_3 = (1 + \nu)(2/\pi r)^{1/2} (K_{\mathrm{I}} \cos \theta/2 - K_{\mathrm{II}} \sin\theta/2) \tag{10}$$

for plane strain conditions.

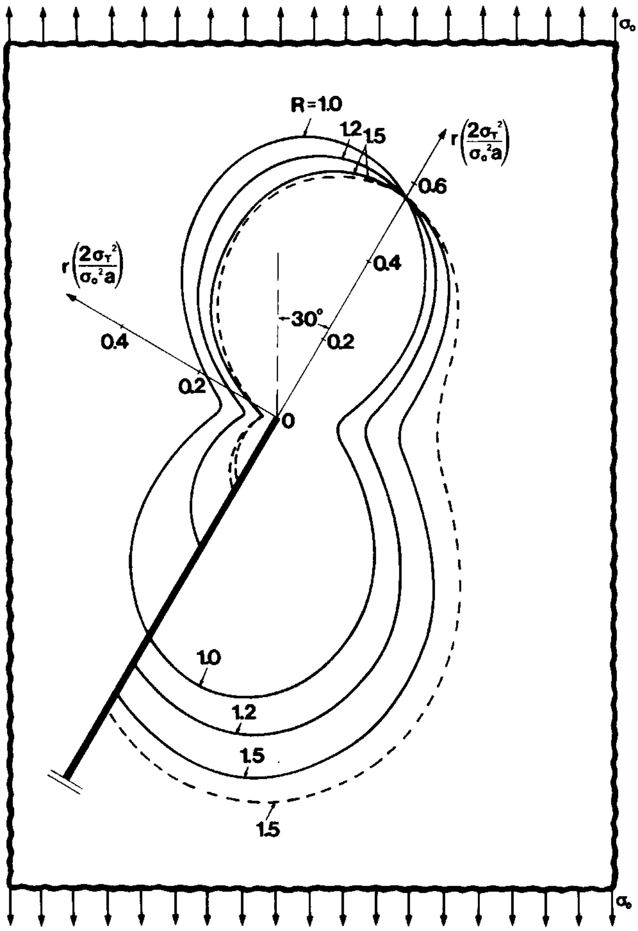

Fig. 4. As in Figure 2 for a cracked plate subjected to plane strain conditions with $\nu = 0.5$.

Furthermore, we obtain

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = (1/\pi r)(a_{11}K_I^2 + 2a_{12}K_IK_{II} + a_{22}K_{II}^2) \tag{11}$$

with

$$\begin{aligned} a_{11} &= \frac{3}{2}\cos^2\frac{\theta}{2}\left(\frac{Q}{3} + \sin^2\frac{\theta}{2}\right) \\ a_{12} &= \frac{3}{4}\left(-\frac{Q}{3} + \cos\theta\right)\sin\theta \\ a_{22} &= \frac{3}{2}\left[1 + \frac{1}{3}\sin^2\frac{\theta}{2}\left(Q - 9\cos^2\frac{\theta}{2}\right)\right] \end{aligned} \tag{12}$$

where the coefficient Q is equal to 1 or $(1 - 2\nu)^2$ for plane stress or plane strain conditions, respectively.

It can be observed from the above relations that the plane stress condition can be obtained from the plane strain by putting $\nu = 0$.

For the determination now of the value r of the elastic–plastic boundary

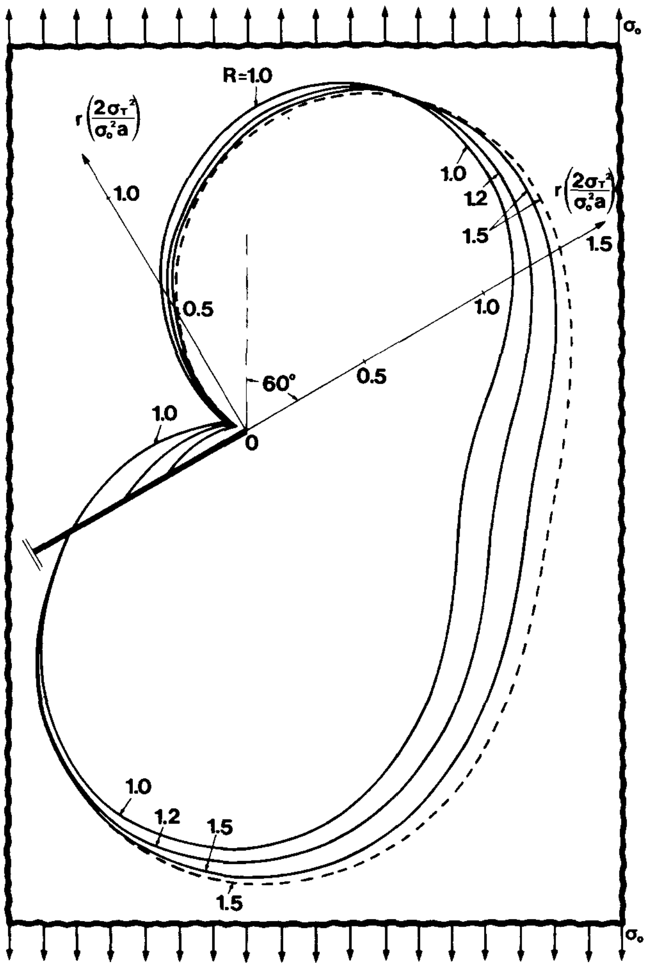

Fig. 5. As in Figure 2 for $\beta = 60°$.

relations (9), (10), (11), and (12) we introduced into relations (3) and (4). In order to have a picture of the evolution of the shape and size of plastic zones for different amounts of K_{I} and K_{II} stress intensity factors, the case of an inclined crack in a plate subjected to uniaxial stress σ_0 at an angle β with respect to the crack axis was considered. For this case the values of K_{I} and K_{II} are given by relations (2).

For the particular case when the yield behavior of the material is described by the von Mises criterion, we obtain for $R = 1$ the following relation for the radius r of the elastic–plastic boundary under plane strain conditions:

$$r = \frac{1}{2\pi\sigma_y^2}\left[K_{\mathrm{I}}^2 \cos^2\frac{\theta}{2}\left[(1-2\nu)^2 + 3\sin^2\frac{\theta}{2}\right] + K_{\mathrm{I}}K_{\mathrm{II}}\sin\theta[3\cos\theta - (1-2\nu)^2]\right.$$
$$\left. + K_{\mathrm{II}}^2\left[3 + \sin^2\frac{\theta}{2}\left\{(1-2\nu)^2 - 9\cos^2\frac{\theta}{2}\right\}\right]\right] \qquad (13)$$

In the following the characteristic features of the elastic–plastic boundary for the case of the inclined crack in an infinite plate under uniaxial tension will be studied in detail.

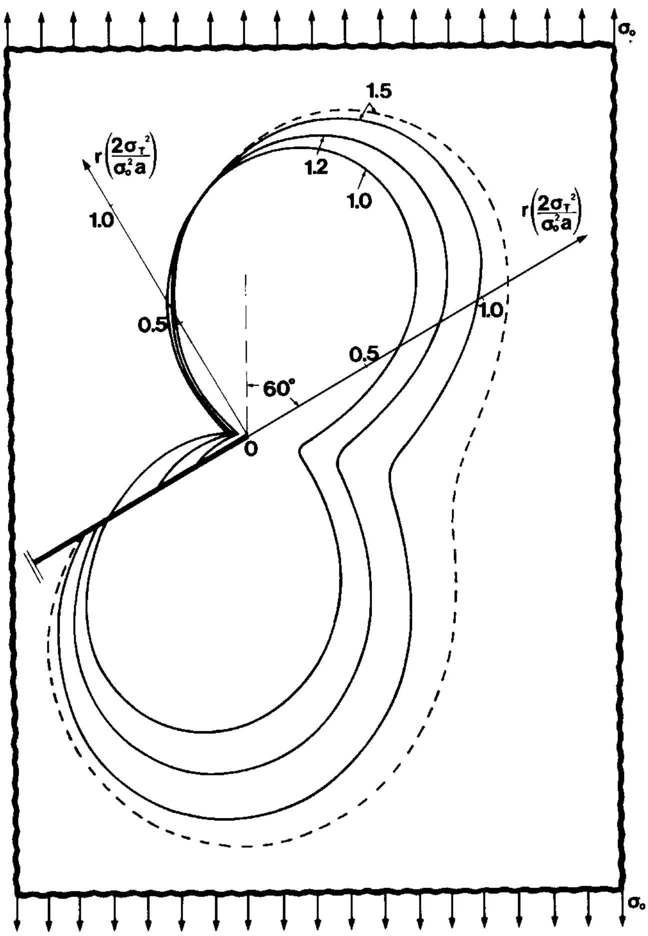

Fig. 6. As in Figure 3 for $\beta = 60°$.

RESULTS

Figures 2, 3, and 4 present the elastic–plastic boundaries around the tip of an inclined crack whose axis makes an angle β equal to 30° with respect to the direction of the applied stress for the values of the Poisson's ratio ν equal to 0, 0.3, and 0.5, respectively. All the curves of the figures correspond to plane strain conditions in the vicinity of the crack tip. The continuous curves of the figures correspond to the PMOSC crierion, while the dotted curves to the PMMC criterion. The constant R expressing the ratio of the compressive to tensile yield stress of the material takes the values 1.0, 1.2, and 1.5 for the PMOSC criterion and the vlaues 1.0 and 1.5 for the PMMC criterion. The elastic–plastic boundaries corresponding to the value $R = 1.2$ for the PMMC criterion were not drawn for the clarity of the figures. The curves $R = 1$ correspond to the von Mises yield criterion. Similarly, Figures 5, 6, and 7 give the elastic–plastic boundaries for $\beta = 60°$ and $\nu = 0, 0.3$, and 0.5 respectively. Note that the case $\nu = 0$ corresponds to plane stress conditions in the region near to the crack-tip. The corresponding curves of Figures 2–7 according to both PMOSC and PMMC criteria for $\beta = 90°$ were given in Ref. 1 and are not repeated here. Furthermore,

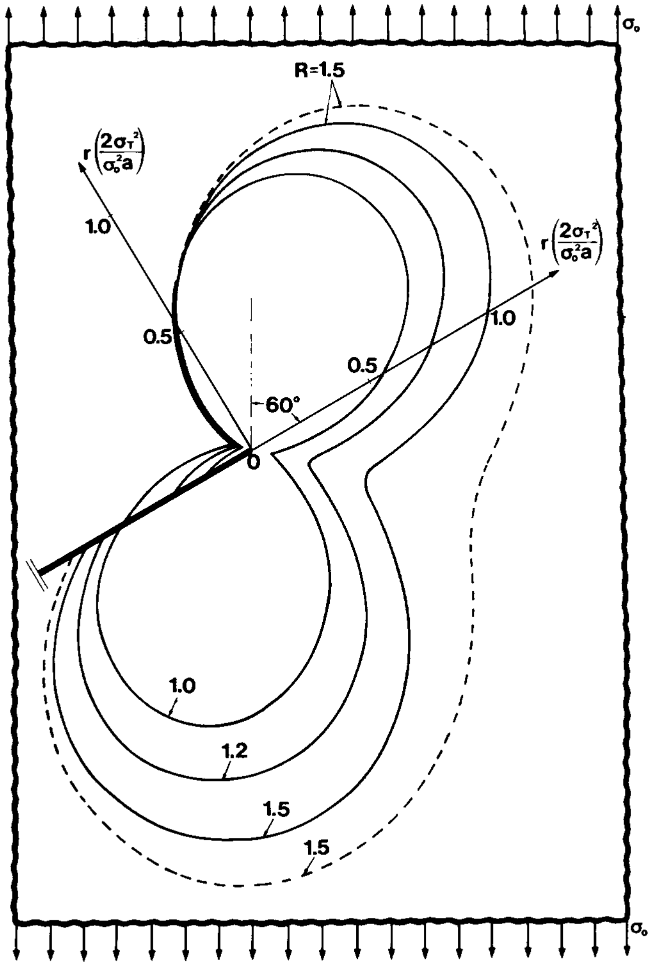

Fig. 7. As in Figure 4 for $\beta = 60°$.

for $\beta = 0$ both K_I and K_{II} stress intensity factors are zero, and therefore this case corresponds to a plate without a crack. Thus, for $\beta = 0$ no plastic zone is formed around the crack tip.

From Figures 2–7 and the corresponding figures for $\beta = 90°$ it can be observed that both PMOSC and PMMC criteria predict different plastic zones from those obtained by the von Mises criterion. The deviation in the size and shape of the plastic zones between the von Mises and the PMOSC and PMMC criteria increases as R increases. As found in Ref. 1 for the case when the applied stress is perpendicular to the crack axis ($\beta = 90°$), the plastic zones obtained by both PMOSC and PMMC criteria are always larger than those obtained by the von Mises criterion. Furthermore, it was found that the PMMC criterion yields larger plastic zones than the PMOSC criterion. Both these rules are not valid for the case when the crack is inclined with respect to the applied stress. As it can be observed from Figures 2–7 the radius of the elastic–plastic boundary predicted by PMOSC and PMMC criteria is larger than that of the von Mises criterion only for the values of the polar angle θ, lying in the interval $-180° < \theta < \theta_{cr}$, where the angle θ_{cr} depends on the value of the crack inclination angle

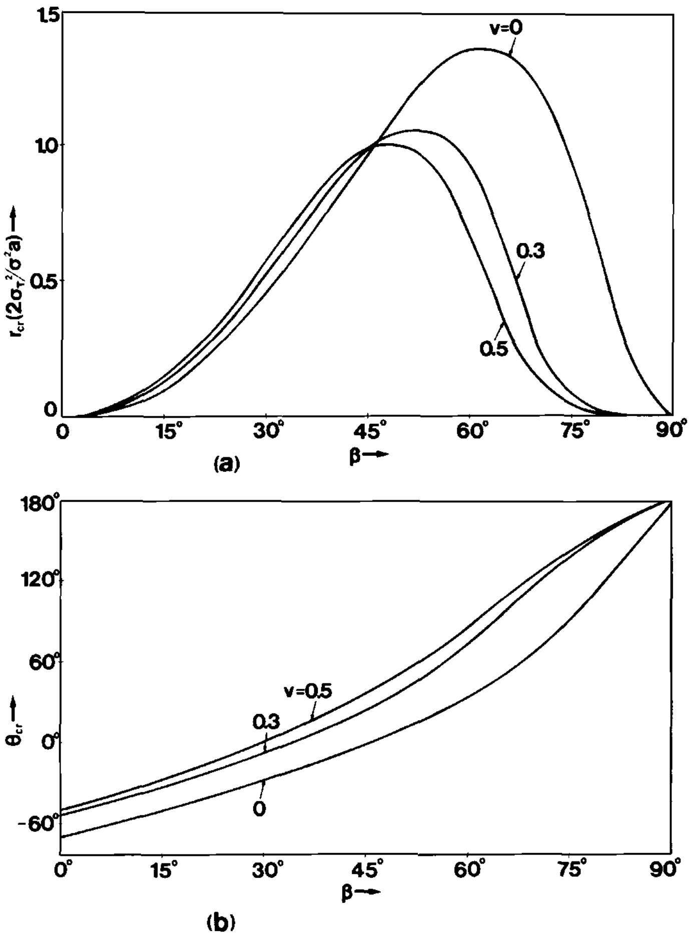

Fig. 8. Variation of the normalized radius $r_{cr}(2\sigma_T^2/\sigma^2 a)$ of the elastic–plastic boundary at $\theta = \theta_{cr}$ (a) and the corresponding angle θ_{cr} (b) vs. the crack inclination angle β for various values of the Poisson's ratio ν. The angle θ_{cr} represents the value of the polar angle θ for which the elastic–plastic boundaries for various values of R intersect at the same point.

β and the Poisson's ratio ν of the plate. For the remaining polar angles $\theta_{cr} < \theta < 180°$ the above rule is reversed and the radius of the elastic–plastic boundary predicted by PMOSC and PMMC criteria is smaller than that obtained by the von Mises criterion. Furthermore, we observe that for the first case ($-180° < \theta < \theta_{cr}$) the PMMC criterion predicts larger elastic–plastic radii than the PMOSC criterion, while for $\theta_{cr} < \theta < 180°$ this rule is reversed.

The critical angle θ_{cr} for which the transition takes place is independent of the value of R and depends only on the values of β and ν. Therefore, as can be observed from Figures 2–7, the elastic–plastic boundaries for each particular value of β and ν pass from the same point for all values of R. Furthermore, we observe that θ_{cr} increases both with β and ν. Thus, for $\beta = 90°$, θ_{cr} becomes equal to 180°, and the plastic zones predicted by PMOSC and PMMC criteria are larger than those of the von Mises criterion. The variation of the normalized radius $r_{cr}(2\sigma_T^2/\sigma^2 a)$ of the elastic–plastic boundary at $\theta = \theta_{cr}$ and the corresponding angle θ_{cr} vs. β for various values of ν is shown in Figures 8(a) and 8(b), respectively.

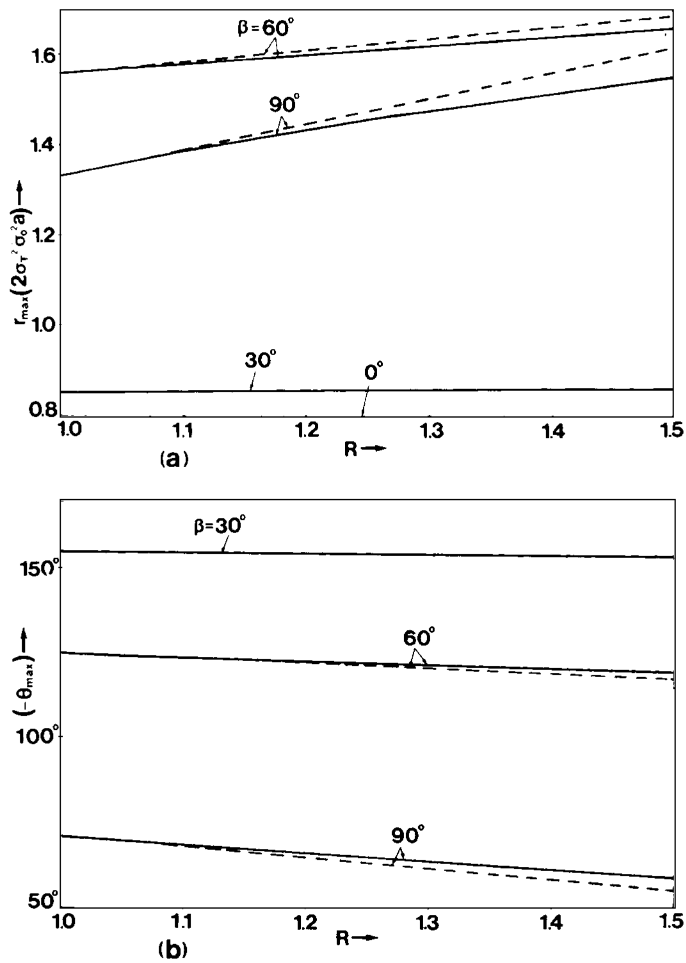

Fig. 9. Variation of the normalized maximum radius $r_{max}(2\sigma_T^2/\sigma_0^2 a)$ of the elastic–plastic boundary and the corresponding angle $(-\theta_{max})$ vs. R for various values of the crack inclination angle β. The plate is subjected to plane stress conditions. The continuous lines correspond to the PMOSC criterion, while the dotted lines to the PMMC criterion. $\nu = 0$.

Another interesting feature of the elastic–plastic boundary for the inclined crack is that while for the case of the von Mises criterion it forms a continuous closed curve which encloses the crack tip, it presents discontinuities along the crack ($\theta = \pm 180°$) for both PMOSC and PMMC criteria. This is shown in all of Figures 2–7, and it is due to the fact that the sum of the principal stresses which enters in both PMOSC and PMMC criteria is discontinuous along the crack. The same phenomenon was also observed[12] in the case when a cracked specimen with an inclined crack was illuminated by a light beam and the obtained caustic on a screen at some distance from the specimen presented the same type of discontinuity along the crack. The caustic forms a highly illuminated curve and is created from the strongly deviated light rays reflected or refracted from the surface of the specimen in the vicinity of the crack tip. The deviation of the light rays is due to the change of the optical path, which depends on the variation of the thickness and the refractive index of the specimen. Thus, the caustics are

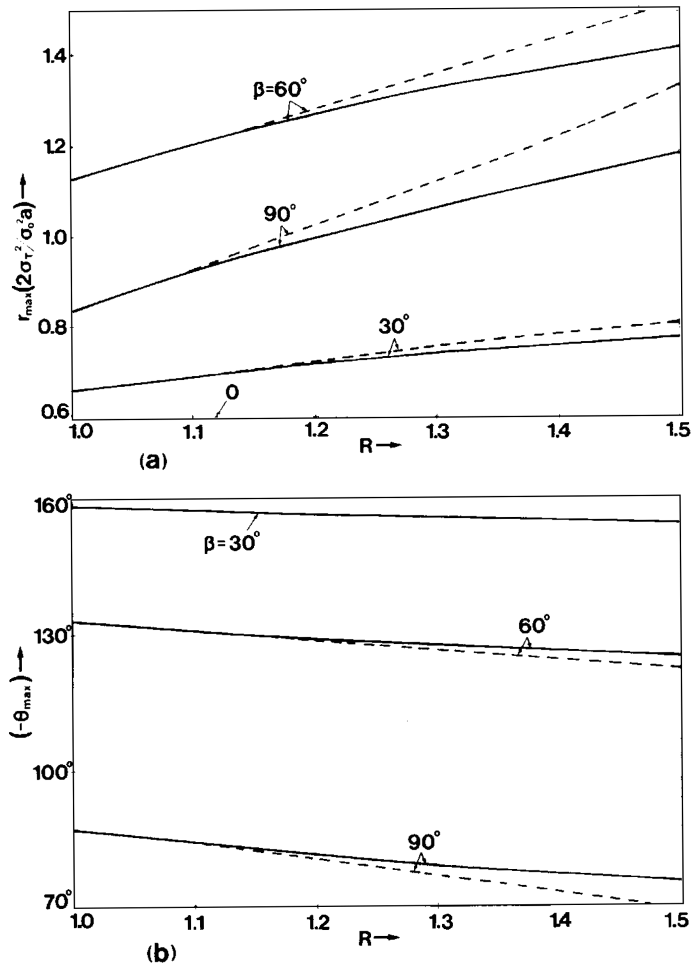

Fig. 10. As in Figure 9 for a cracked plate subjected to plane strain conditions with $\nu = 0.3$.

closely related to the sum of the principal stresses, and their discontinuity along the crack is due to the same reason as in our case. The characteristic discontinuity of the caustics along the crack has extensively been used for the study of the stress field around cracks subjected to mixed mode (opening-mode and sliding-mode) loading conditions.

A main characteristic element of the elastic–plastic boundary around the tip of a crack is its maximum radius r_{max} occurring at a definite direction dictated by the angle θ_{max}. The variation of the quantities $r_{max}(2\sigma_T^2/\sigma_0^2 a)$ and $(-\theta_{max})$ vs. R for various values of the crack inclination angle β is presented in Figures 9, 10, and 11 for $\nu = 0, 0.3$, and 0.5, respectively. The continuous lines of these figures correspond to the PMOSC criterion, while the dotted lines to the PMMC criterion. It is observed that r_{max} increases with R and decreases with ν. Furthermore, we observe that the deviation of the predictions of PMOSC and PMMC criteria increases when the crack inclination angle β also increases. Thus, the largest deviation takes place for the case when the crack is perpendicular to the applied stress. Also the deviation in the predictions of both criteria increases with the Poisson's ratio ν.

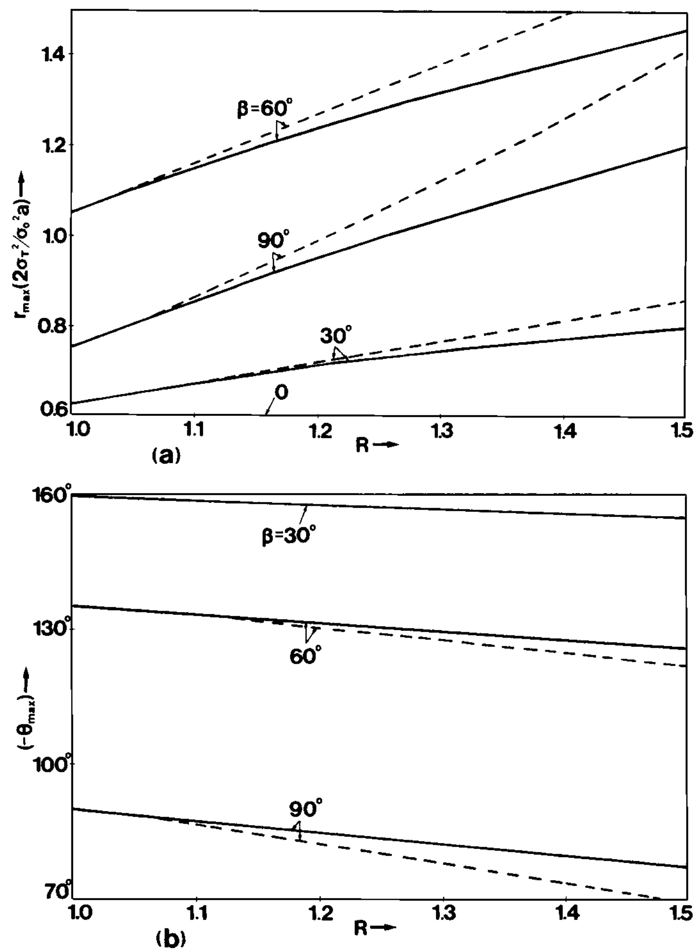

Fig. 11. As in Figure 9 for a cracked plate subjected to plane strain conditions with $\nu = 0.5$.

CONCLUSIONS

The plastic zones developed around the tip of a crack in a plate subjected to combined opening-mode and sliding-mode loading conditions were studied. The material of the plate was a glassy polymer whose yield behavior depends on the hydrostatic component of the stress tensor and which is characterized by different values of the yield stresses in tension and compression. Two pressure-modified von Mises yield criteria which take into account both these characteristic properties of glassy polymers were used. The shape and size of the plastic zones and their dependence on the crack inclination angle, the Poisson's ratio, and the ratio of the compressive to tensile yield stresses of the plate were investigated. From the whole study the following results may be derived:

(i) The pressure-modified yield criteria predict different plastic zones from those obtained by the von Mises criterion.

(ii) The deviation in the size and shape of the plastic zones determined by the pressure-modified and the von Mises criterion increases with the ratio of the compressive to tensile yield stresses of the material as well as with the Poisson's ratio.

(iii) The radius of the elastic–plastic boundary determined by the PMOSC and PMMC criteria is larger than that obtained by the von Mises criterion for all values of the polar angle lying in the interval $-180° < \theta < \theta_{cr}$ and smaller for the remaining angles. The angle θ_{cr} is independent of the ratio of the compressive to tensile yield stress and increases with the crack inclination angle and the Poisson's ratio.

(iv) The elastic–plastic boundary developed around an inclined crack and determined by the pressure modified yield criteria does not form a closed curve but presents discontinuities along the crack.

(v) The maximum radius of the plastic zone increases as the ratio of the compressive to tensile yield stress increases, and it decreases with the Poisson's ratio of the material. The difference in the predictions of the maximum radius by the two pressure modified criteria increases with the crack inclination angle, the ratio of the compressive to tensile yield stress, and the Poisson's ratio.

References

1. E. E. Gdoutos, *J. Appl. Polym. Sci.*, **26**, 1919 (1981).
2. L. P. Pook, *Eng. Fract. Mech.*, **3**, 205 (1971).
3. P. S. Theocaris, *Int. J. Mech. Sci.*, **18**, 171 (1976).
4. A. S. Argon, *Plastic Deformation in Glassy Polymers*, ASM Seminar, Chicago, Sept. 29–30, 1973, pp. 1–59.
5. P. C. Paris and G. C. Sih, "Fracture Toughness Testing and Its Applications," ASTM STP 381, 1964, pp. 30–81.
6. A. Nadai, *Theory of Flow and Fracture of Solids*, McGraw-Hill, New York, 1950, pp. 210, 220.
7. J. C. Bauwens, *J. Polym. Sci. A2*, **5**, 1145 (1967).
8. J. C. Bauwens, *J. Polym. Sci. A2*, **8**, 893 (1970).
9. S. S. Sternstein and L. Ongchin, *Am. Chem. Soc. Polym.*, **10**, 1117 (1969).
10. R. S. Raghava and R. M. Caddell, *Int. J. Mech. Sci.*, **15**, 967 (1973).
11. R. S. Raghava, R. M. Caddell, and G. S. Y. Yeh, *J. Mat. Sci.*, **8**, 225 (1973).
12. P. S. Theocaris and E. E. Gdoutos, *J. Appl. Mech.*, **39**, 91 (1972).

Received May 18, 1981
Accepted June 30, 1981
Corrected proofs received January 11, 1982